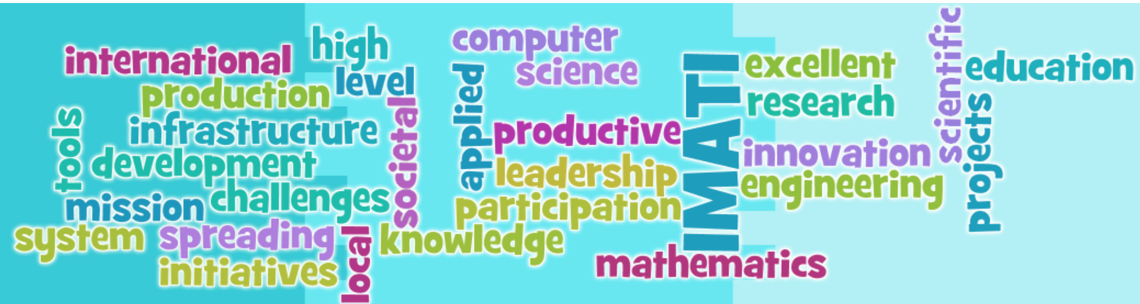
Topological Data Analysis

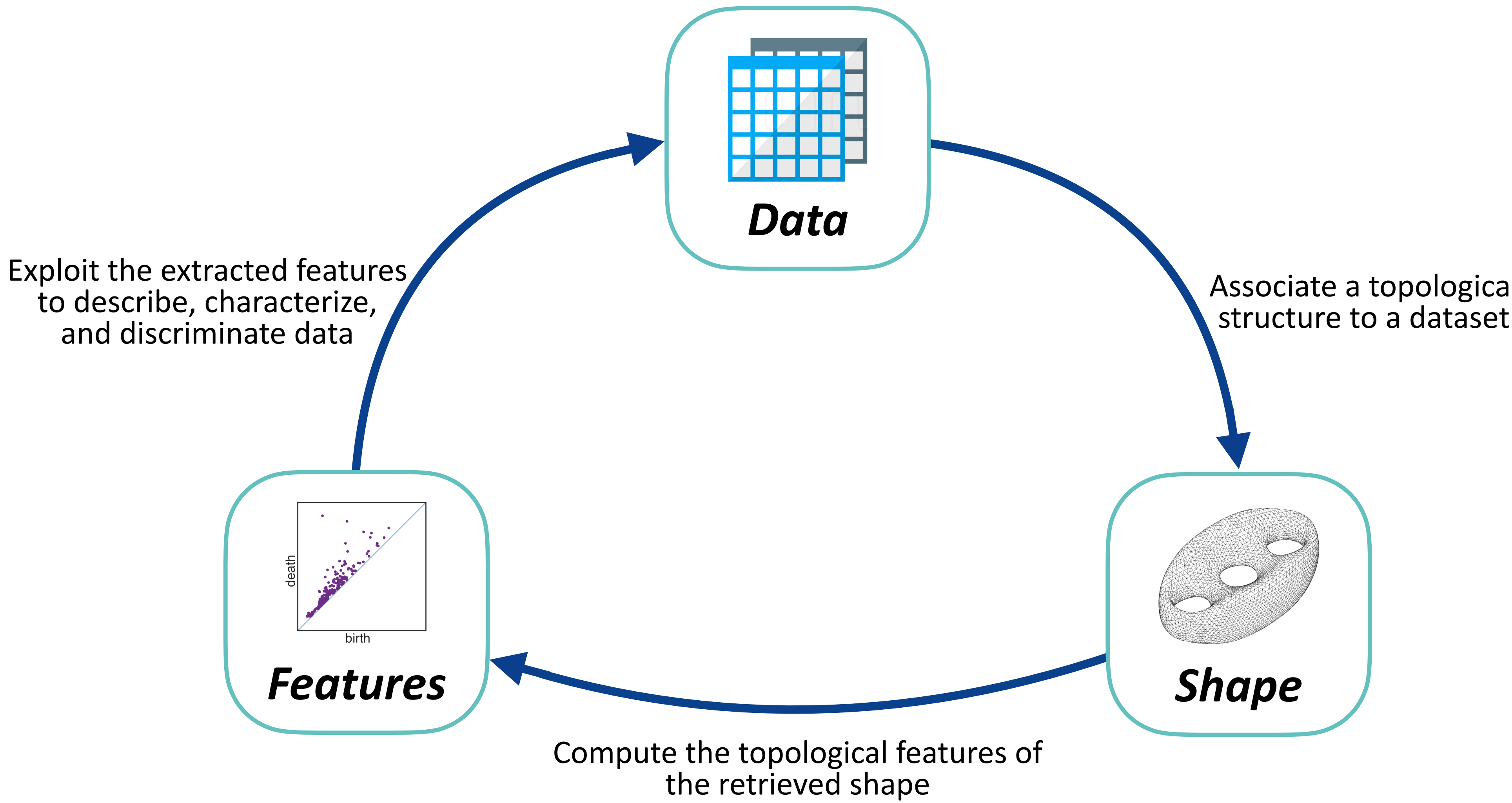
Morse Theories

Ulderico Fugacci

CNR - IMATI



Topological Data Analysis



Topological Data Analysis

Tools in TDA:

- **Elementary Tools**:
  - Euler Characteristic
  - Reeb Graph
  - Mapper
- **Homology-Based Tools**:
  - Homology
  - Directed Homology
  - Persistent Homology
  - Zigzag Persistence
  - Multi-Parameter Persistent Homology
- **Other Tools**:
  - Morse Theories

Morse Theories

- *(Smooth) Morse Theory*
- *Discrete Morse Theory*

Morse Theories

- *(Smooth) Morse Theory*
- *Discrete Morse Theory*

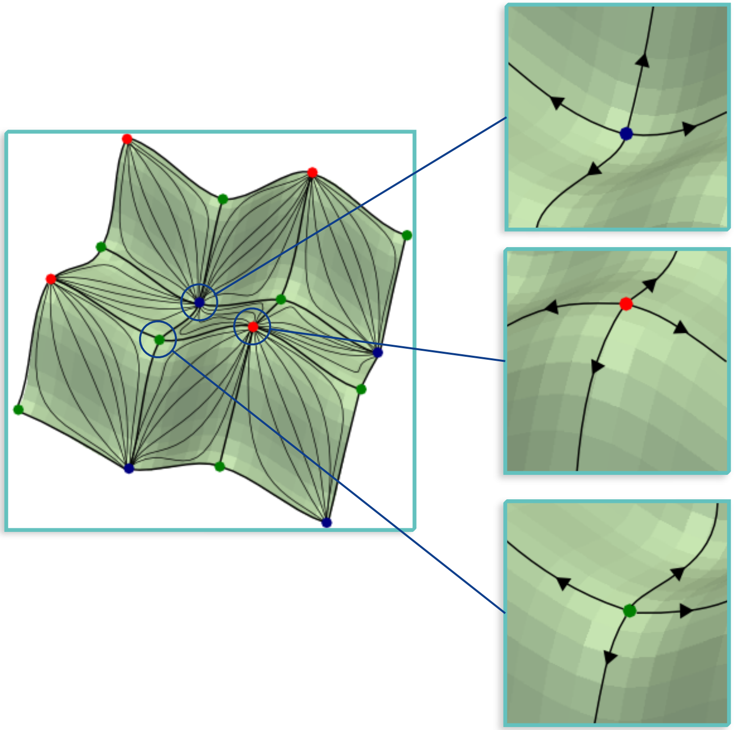
Morse Theory

Morse Theory ***[Milnor 1963, Matsumoto 2002] :***

- Topological tool for efficiently analyzing a ***shape of a data*** by studying the behavior of a smooth ***scalar function f*** defined on it

- Relates the critical points of a smooth scalar function on a shape with their ***regions of influence***

- Analysis of scalar fields requires extracting ***morphological features*** (e.g., critical points, integral lines and surfaces)



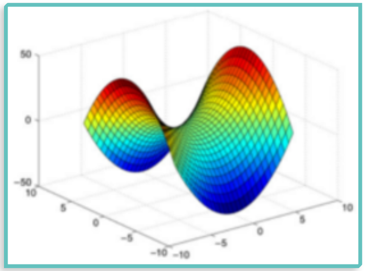
Morse Theory

Critical Points:

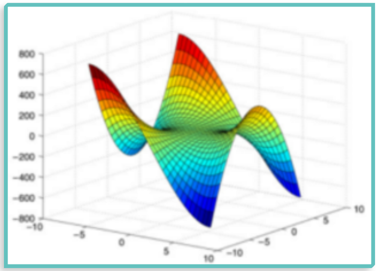
Let f be a real-valued C^2-function defined on a d-dimensional manifold M

- ***Critical point*** of f:
  any point on M in which the gradient of f vanishes

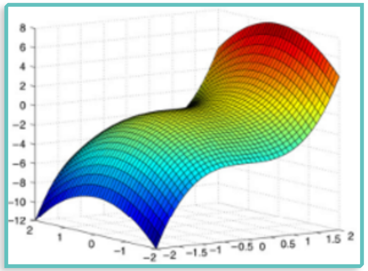
- Critical points can be ***degenerate*** or ***non-degenerate***
  - A critical point p is degenerate *iff* the determinant the Hessian matrix H of the second order derivatives of function f is null



non-degenerate critical point



degenerate critical point



degenerate critical point

Morse Theory

Critical Points:

Let f be a real-valued C^2-function defined on a d-dimensional manifold M

- ***Critical point*** of f:
  any point on M in which the gradient of f vanishes

- Critical points can be ***degenerate*** or ***non-degenerate***
  - A critical point p is degenerate *iff* the determinant the Hessian matrix H of the second order derivatives of function f is null

*Function f is a **Morse function** if and only if*

all its critical points are non-degenerate

Morse Theory

Definitions:

Gradient

$$\nabla f = \frac{\partial f}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial f}{\partial x_n}\mathbf{e}_n$$

Hessian Matrix

$$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1\,\partial x_n} \\ \frac{\partial^2 f}{\partial x_2\,\partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n\,\partial x_1} & \frac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Morse Theory

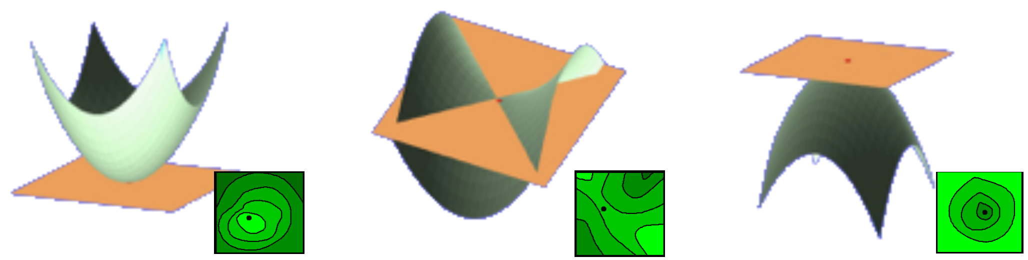
Critical Points:

Let f be a real-valued C^2-function defined on a d-dimensional manifold M

- Critical points of a Morse function are ***isolated***

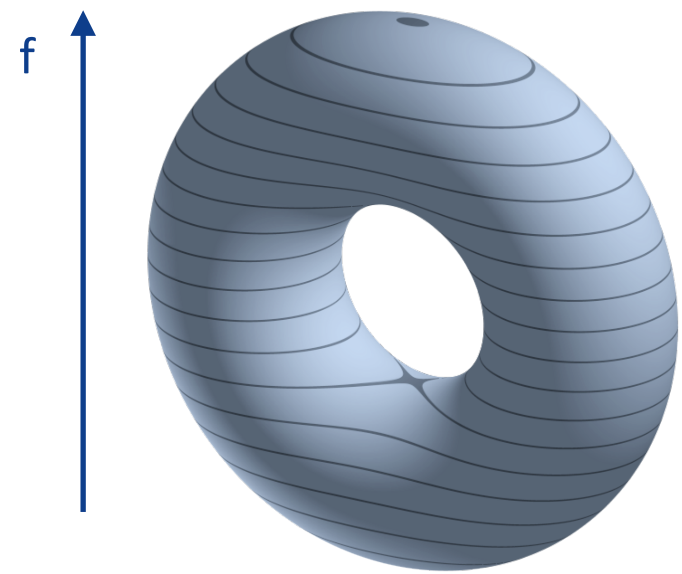
- A d-dimensional Morse function f has d+1 types of critical points, called ***k-saddles***
  (k is the *index* of the critical point)
  - For d=2, ***minima***, ***saddles*** and ***maxima***
  - For d=3, ***minima***, ***1-saddles***, ***2-saddles*** and ***maxima***

Critical points of a 2D function



Morse Theory

Fundamental Theorems:



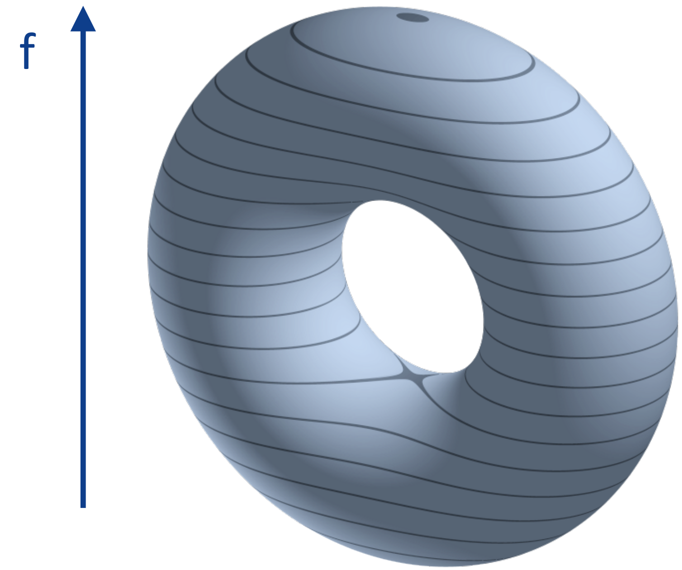
Theorem 1:

Suppose f is a smooth real-valued function on M, a < b, $f^{-1}[a, b]$ is compact, and there are no critical values between a and b.

*Then, M^a is **diffeomorphic** to M^b.*

Morse Theory

Fundamental Theorems:

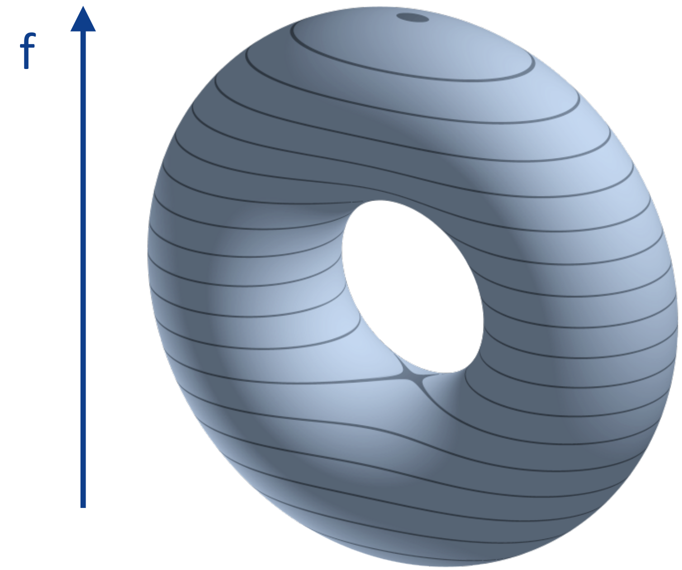


Theorem 2:

Suppose f is a smooth real-valued function on M and
p is a non-degenerate critical point of f of index k, and that $f(p) = q$.
Suppose $f^{-1}[q - \varepsilon, q + \varepsilon]$ is compact and contains no critical points besides p.
Then, $M^{q+\varepsilon}$ is ***homotopy equivalent*** *to $M^{q-\varepsilon}$ with a k-cell attached.*

Morse Theory

Fundamental Theorems:



Morse Inequalities:

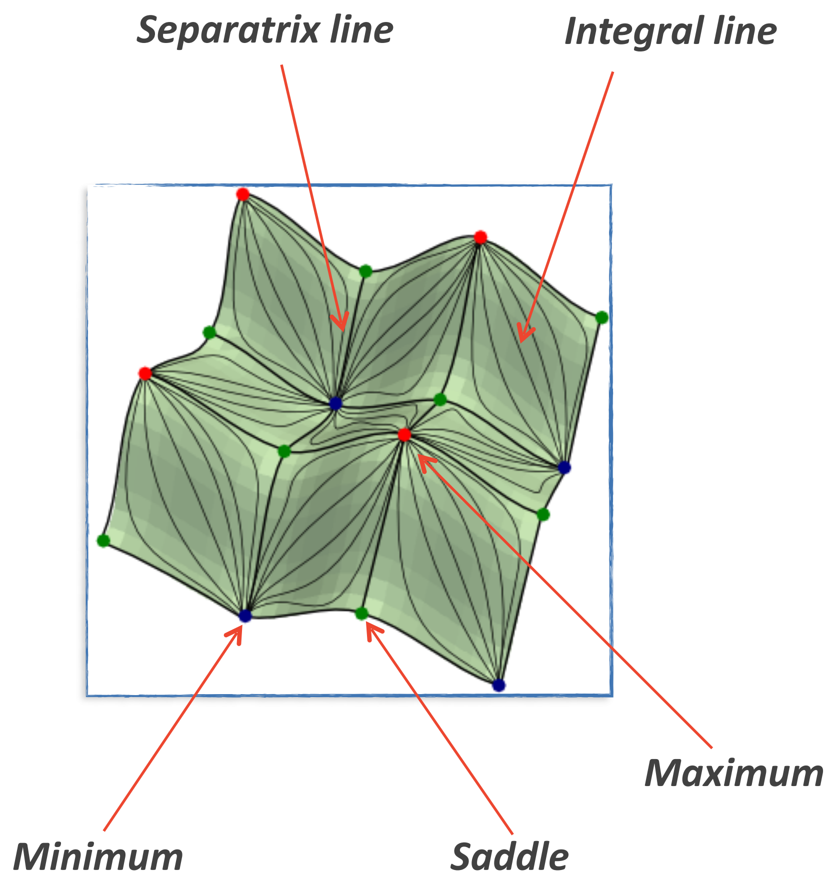
$$c_k \geq \beta_k(M)$$

where c_k is the number of critical points of index k

Morse Theory

Integral Lines:

- An ***integral line*** of a smooth function f is a maximal path which is everywhere tangent to the gradient vector field of f
- Integral lines ***start*** and ***end*** at the critical points of f
- Integral lines that connect critical points of consecutive index are called ***separatrix lines***



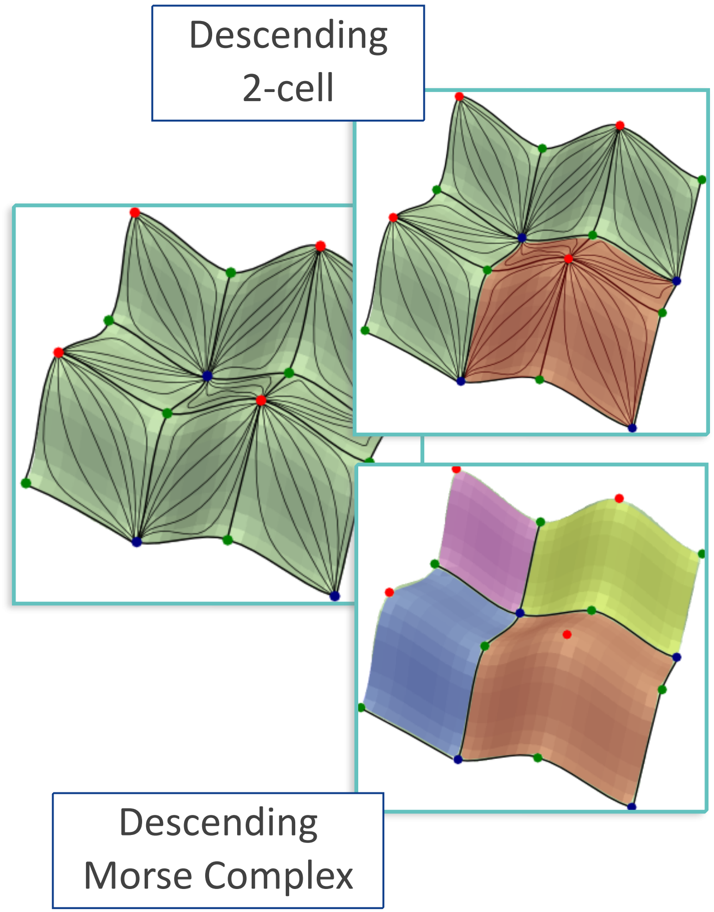
Morse Theory

Integral lines that converge to a critical point p of index i form an i-cell called the ***descending cell*** of p

- Descending cell of a maximum: 2-cell
- Descending cell of a saddle: 1-cell
- Descending cell of a minimum: 0-cell

Descending Morse Complex:

Collection of the descending cells of all critical points of function f



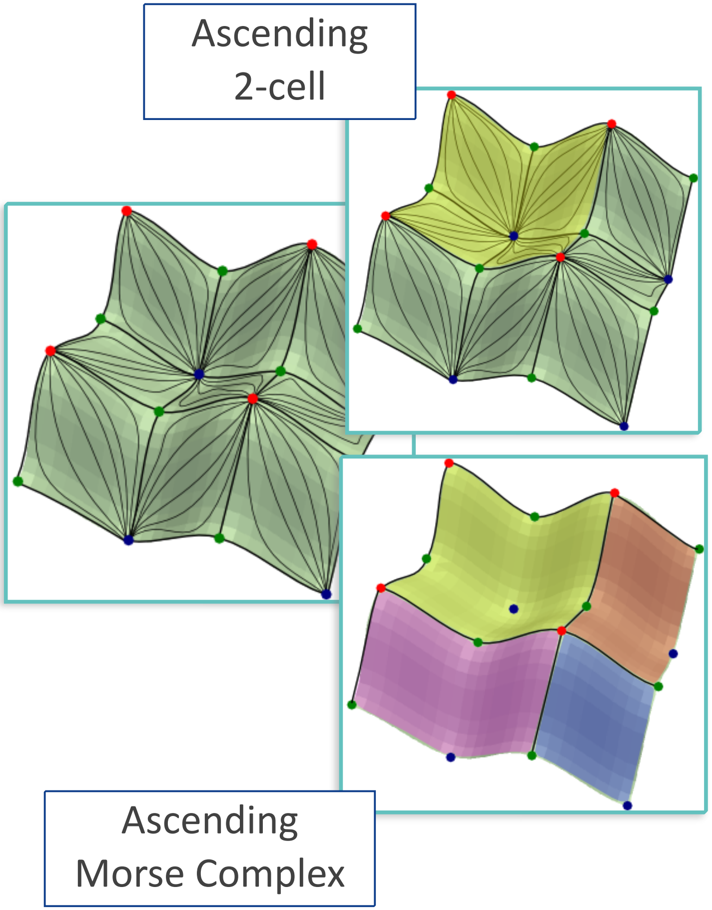
Morse Theory

Integral lines that converge to a critical point p of index i form a (d-i)-cell called the ***ascending cell*** of p

- Ascending cell of a minimum: 2-cell
- Ascending cell of a saddle: 1-cell
- Ascending cell of a maximum: 0-cell

Ascending Morse Complex:

Collection of the ascending cells of all critical points of function f

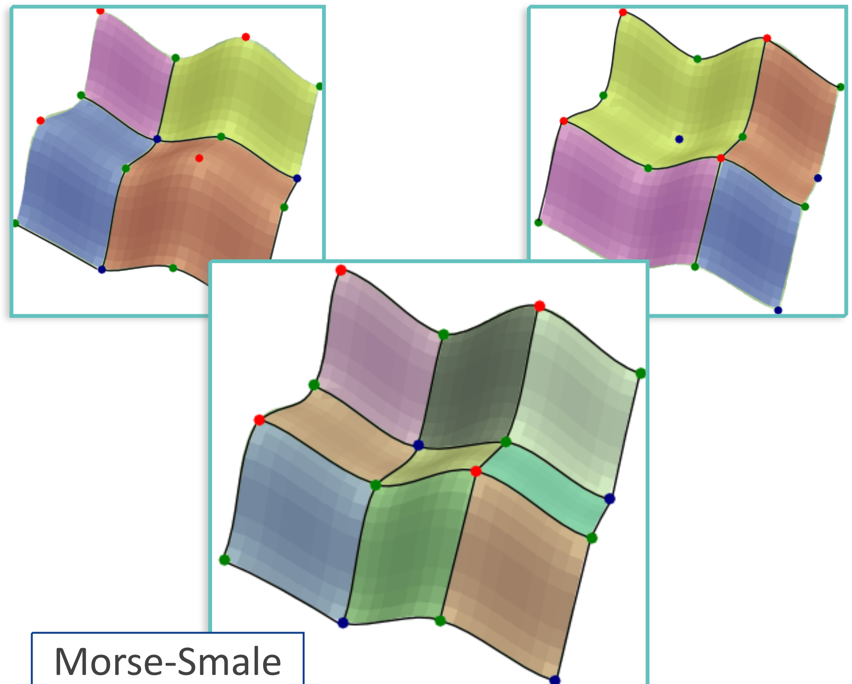


Morse Theory

Morse Smale Complex:

- Function f is a ***Morse-Smale function*** if its ascending and descending Morse cells intersect transversally

- ***Morse-Smale (MS) complex*** is the complex obtained from the mutual intersection of all the ascending and descending cells



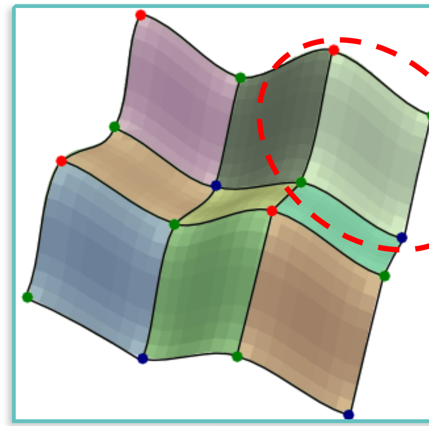
Morse-Smale complex

Morse Theory

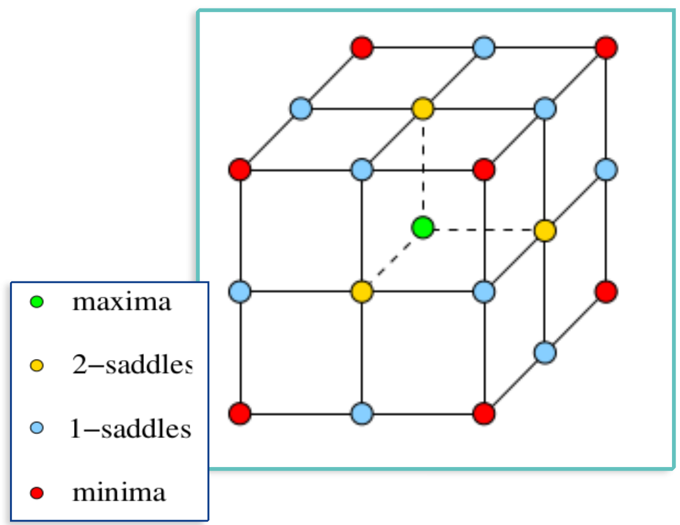
Morse Smale Complex:

- In a ***2D*** Morse-Smale complex:
  - A 2-cell is a ***quadrilateral*** bounded by the sequence

    maximum – saddle – minimum – saddle



- In a ***3D*** Morse-Smale complex:
  - Each 1-saddle is connected to exactly two minima
  - Each 2-saddle is connected to exactly two maxima



Morse Theory

Applications:

- ***Shape Segmentation***
  - ***Segmenting the boundary of a 3D shape***
  - Volume data segmentation
  - Multi-resolution terrain analysis
  - Multi-resolution analysis of volume data

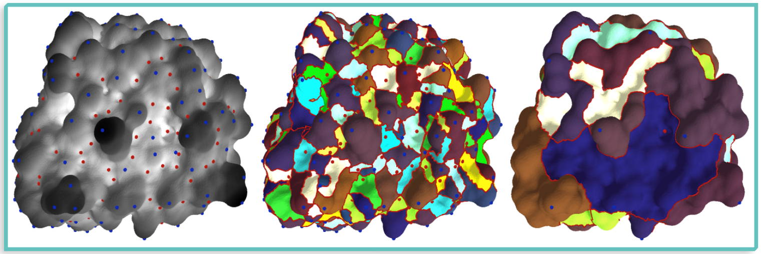


Image from [Natarajan et al. 2006]

Study of cavities and protrusions in an atomic density function

- ***Homological Analysis***
  - Homology computation
  - 3D and higher-dimensional shapes
  - Shapes discretized as simplicial complexes
  - Not only shapes defined by point data, but also networks

Morse Theory

Applications:

- ***Shape Segmentation***
  - ***Segmenting the boundary of a 3D shape***
  - Volume data segmentation
  - Multi-resolution terrain analysis
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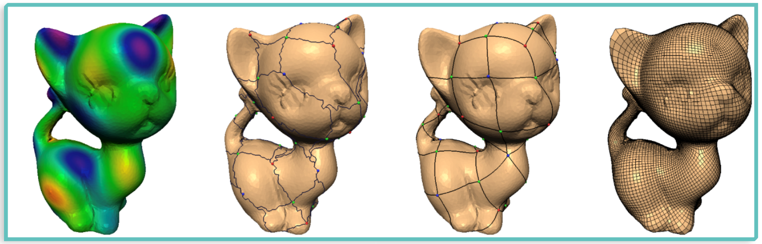

Image from [Dong et al. 2006]

Quad mesh generation from a triangle mesh

- ***Homological Analysis***
  - Homology computation
  - 3D and higher-dimensional shapes
  - Shapes discretized as simplicial complexes
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Morse Theory

Applications:

- ***Shape Segmentation***
  - Segmenting the boundary of a 3D shape
  - ***Volume data segmentation***
  - Multi-resolution terrain analysis
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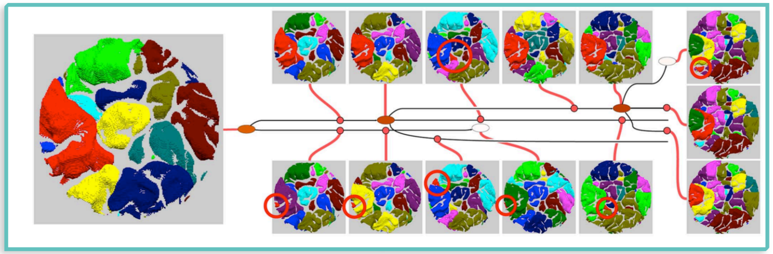

Image from [Bremer et al. 2010]

Burning cells tracked over time
Morse complexes at different time step

- ***Homological Analysis***
  - Homology computation
  - 3D and higher-dimensional shapes
  - Shapes discretized as simplicial complexes
  - Not only shapes defined by point data, but also networks

Morse Theory

Applications:

- ***Shape Segmentation***
  - Segmenting the boundary of a 3D shape
  - Volume data segmentation
  - ***Multi-resolution terrain analysis***
  - Multi-resolution analysis of volume data
- ***Homological Analysis***
  - Homology computation
  - 3D and higher-dimensional shapes
  - Shapes discretized as simplicial complexes
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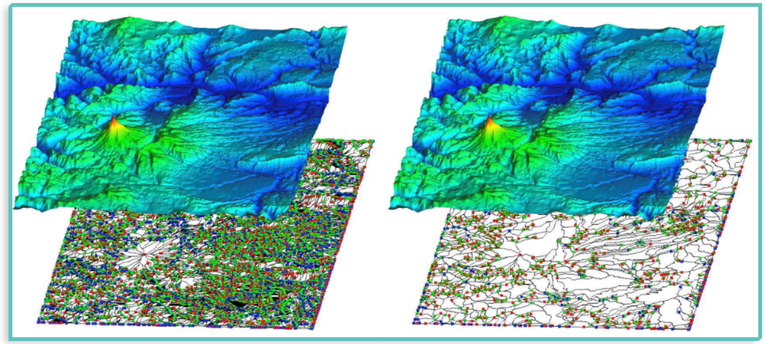


Image from [Bremer et al. 2004]

Network of the critical points at two levels of resolution

Morse Theory

Applications:

- ***Shape Segmentation***
  - Segmenting the boundary of a 3D shape
  - Volume data segmentation
  - Multi-resolution terrain analysis
  - ***Multi-resolution analysis of volume data***

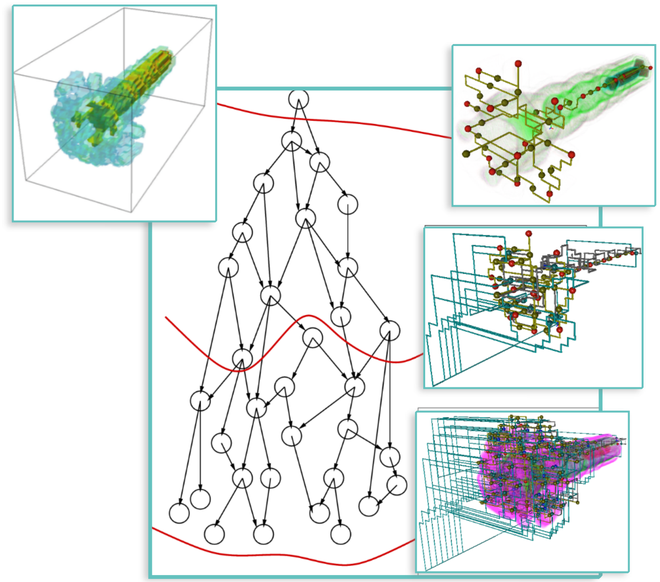


Image from [Gyulassy et al. 2010]

Network of the critical points on a volume data set at different resolution

- ***Homological Analysis***
  - Homology computation
  - 3D and higher-dimensional shapes
  - Shapes discretized as simplicial complexes
  - Not only shapes defined by point data, but also networks

Morse Theory

Applications:

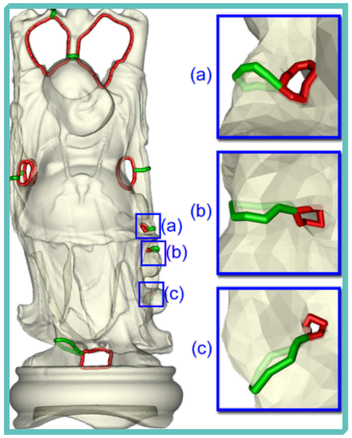


Image from [Dey et al. 2008]

- **Shape Segmentation**
  - Segmenting the boundary of a 3D shape
  - Volume data segmentation
  - Multi-resolution terrain analysis
  - Multi-resolution analysis of volume data

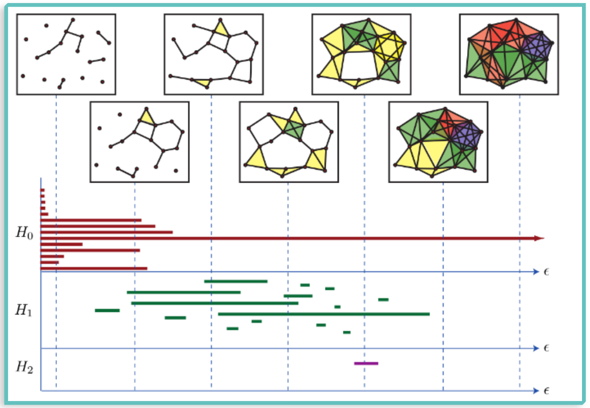


Image from [Ghrist 2008]

- **Homological Analysis**
  - ***Homology computation***
  - ***3D and higher-dimensional shapes***
  - ***Shapes discretized as simplicial complexes***
  - ***Not only shapes defined by point data, but also networks***

Morse Theories

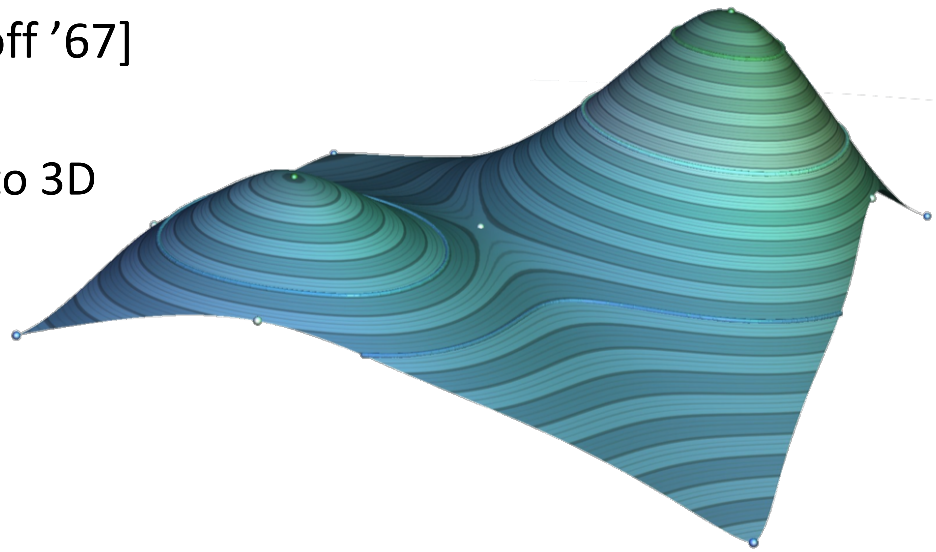
- *(Smooth) Morse Theory*
- ***Discrete Morse Theory***

Discrete Morse Theory

Discretized Morse Theories:

Various *discretizations* of Morse theory:

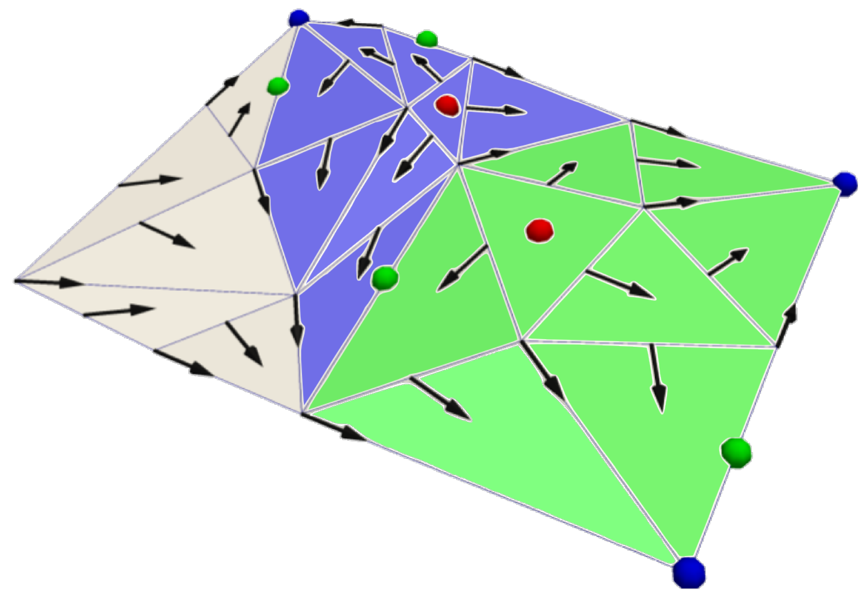
- ***Piecewise-Linear Morse Theory*** [Banchoff '67]
  - Originally for polyhedral surfaces
  - Defined for the 2D case and extended to 3D
- ***Watershed Transform*** [Meyer '94]
  - For images and labeled graphs
  - Dimension-independent
- ***Discrete Morse Theory*** [Forman '98]
  - For cell complexes
  - Dimension-independent



Discrete Morse Theory

Discrete Morse Theory:

Is a ***combinatorial counterpart*** of Morse Theory:

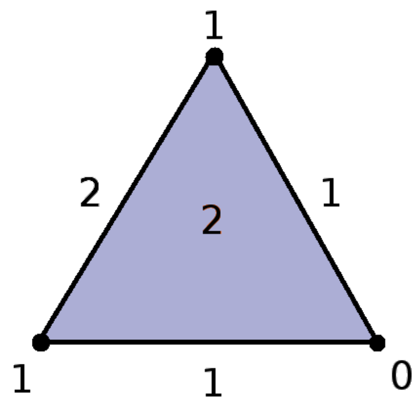


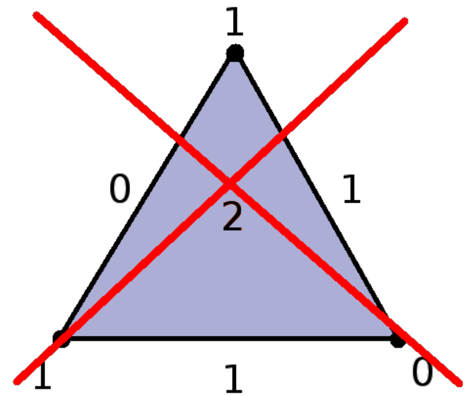
- Introduced for ***cell complexes***
- Providing a compact ***homology-equivalent*** model for a shape
- Representing a ***derivative-free*** tool for computing segmentations of shapes

Discrete Morse Theory

Discrete Morse Function:

Let K be a simplicial complex





$f : K \rightarrow \mathbb{R}$ is called ***discrete Morse function*** if, for every simplex σ,

$$c_+(\sigma) := \#\{\, \tau > \sigma \mid f(\tau) \leq f(\sigma) \,\} \leq 1$$

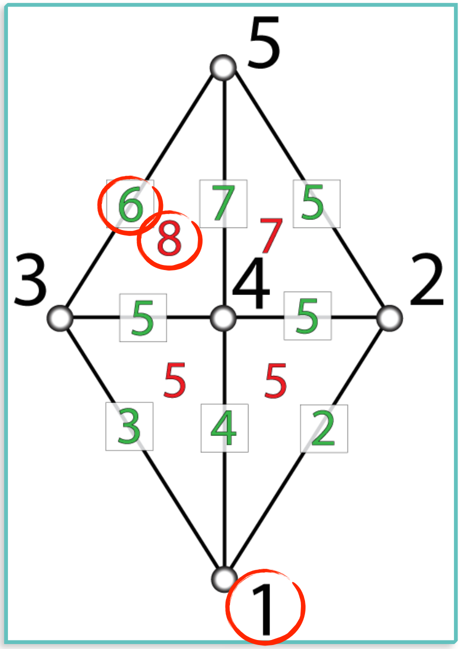
$$c_-(\sigma) := \#\{\, \rho < \sigma \mid f(\rho) \geq f(\sigma) \,\} \leq 1$$

Discrete Morse Theory

Discrete Morse Function:

Proposition:

$c_{+}(\sigma)$ and $c_{-}(\sigma)$ cannot both be equal to 1



Critical Simplices:

A k-simplex σ is ***critical*** *with index k if*

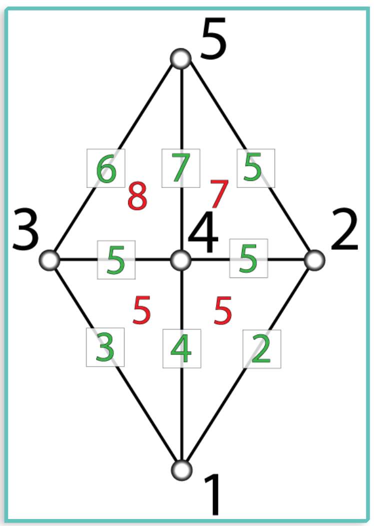
$$\#\{ \tau > \sigma \mid f(\tau) \leq f(\sigma) \} = \#\{ \rho < \sigma \mid f(\rho) \geq f(\sigma) \} = 0$$

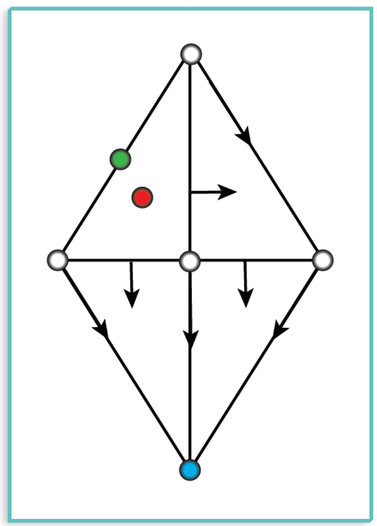
Discrete Morse Theory

Discrete Vector Field:

A collection V of pairs (σ,τ) ∊ K x K such that:

- *σ < τ (i.e.,* ***incident*** *simplices of dimension k and k+1)*
- *Each simplex of K is* ***in at most one pair*** *of V*





Unpaired *simplices*

↕

Critical *simplices*

A discrete Morse function f induces a discrete vector field called the ***gradient vector field*** *of f*

$$V := \{ (\sigma, \tau) \in K \times K \mid \sigma < \tau \text{ and } f(\sigma) \geq f(\tau) \}$$

Discrete Morse Theory

V-Path:

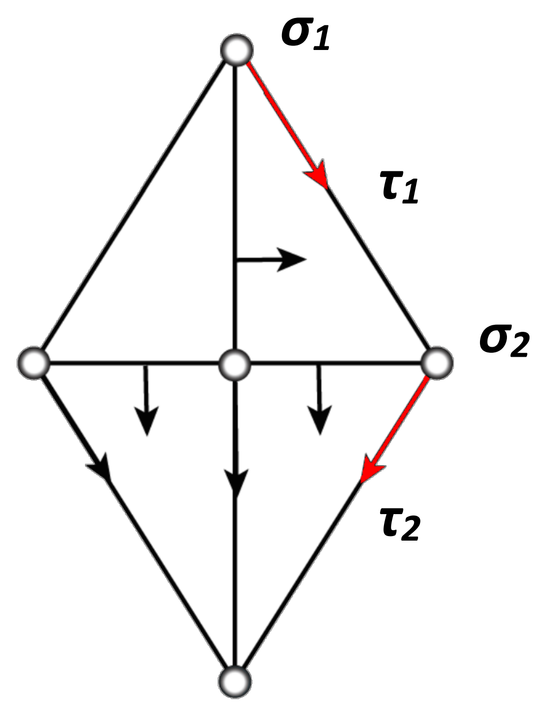
A sequence of pairs of V

$$(\sigma_1,\tau_1), (\sigma_2,\tau_2), \ldots, (\sigma_{r-1},\tau_{r-1}), (\sigma_r,\tau_r)$$

such that:

- $\sigma_{i+1} < \tau_i$
- $\sigma_{i+1} \neq \sigma_i$

*A V-path is **closed** if σ_1 is a face of τ_r different from σ_r*



Proposition:

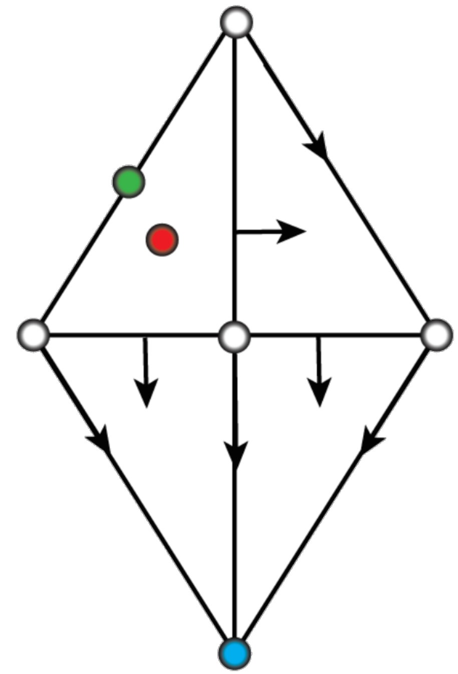
*A discrete vector field V is the gradient vector field of a discrete Morse function if and only if it is **free of closed V-paths***

Discrete Morse Theory

Discrete Morse Complex:

A chain complex whose:

- ***k-cells*** *are in correspondence with critical simplices of index k*
- ***boundary relations*** *are induced by V-paths*

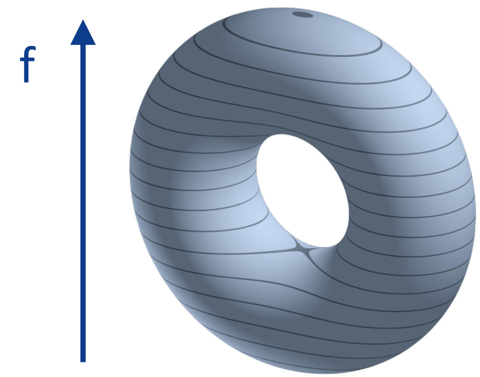


Theorem:

Given a gradient vector field V defined on a simplicial complex K, the associated ***discrete Morse complex is homotopy equivalent to K***

Discrete Morse Theory

Fundamental Theorems:



Smooth Theorem 1:

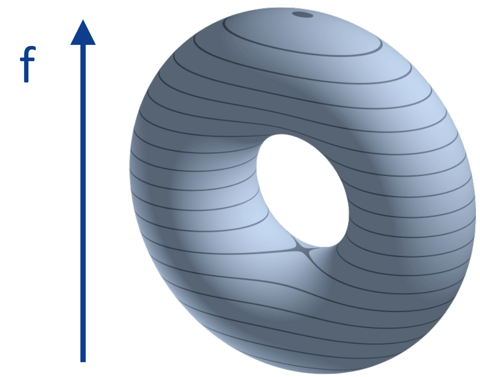
*Suppose f is a smooth real-valued function on M, $a < b$, $f^{-1}[a, b]$ is compact, and there are no critical values between a and b. Then, M^a is **diffeomorphic** to M^b.*

Discrete Theorem 1:

*Suppose f is a discrete Morse function on M, $a < b$, and there are no critical values between a and b. Then, M^a is a **deformation retract** of M^b.*

Discrete Morse Theory

Fundamental Theorems:



Smooth Theorem 2:

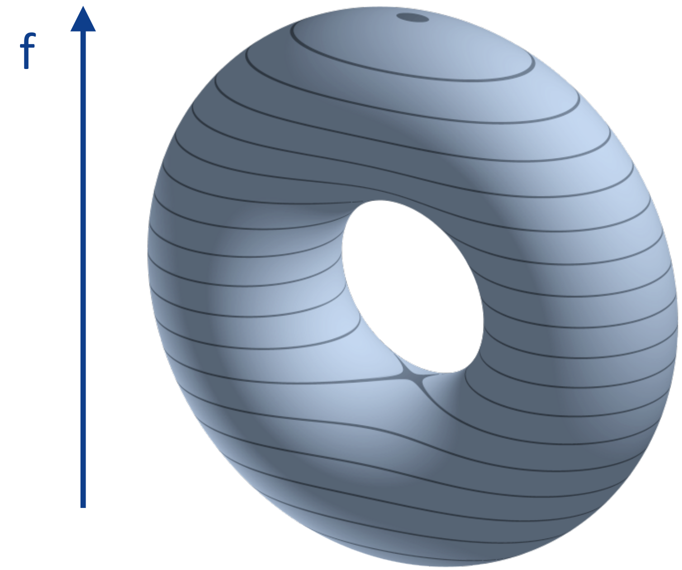
Suppose f is a smooth real-valued function on M and p is a non-degenerate critical point of f of index k, and that f(p) = q. Suppose $f^{-1}[q-\varepsilon, q+\varepsilon]$ is compact and contains no critical points besides p. Then, $M^{q+\varepsilon}$ is ***homotopy equivalent*** *to $M^{q-\varepsilon}$ with a k-cell attached.*

Discrete Theorem 2:

Suppose f is a discrete Morse function on M, σ is a critical k-simplex with $f(\sigma)\in[a, b]$, and there are no other critical simplices with values in [a, b].
Then, M^a is ***homotopy equivalent*** *to M^b with a k-cell attached.*

Discrete Morse Theory

Fundamental Theorems:



Smooth & Discrete Morse Inequalities:

$$c_k \geq \beta_k(M)$$

where c_k is the number of critical points/simplices of index k

Bibliography

General References:

- ***Books on TDA:***
  - A. J. Zomorodian. ***Topology for computing*****.** Cambridge University Press, 2005.
  - H. Edelsbrunner, J. Harer. ***Computational topology: an introduction***. American Mathematical Society, 2010.
  - R. W. Ghrist. ***Elementary applied topology***. Seattle: Createspace, 2014.
- ***Papers on TDA:***
  - G. Carlsson. ***Topology and data***. Bulletin of the American Mathematical Society 46.2, pages 255-308, 2009.

Today's References:

- ***Morse Theories:***
  - L. De Floriani, U. Fugacci, F. Iuricich, P. Magillo. ***Morse complexes for shape segmentation and homological analysis: discrete models and algorithms.*** Computer Graphics Forum 34.2, pages 761-785, 2015.
  - L. De Floriani, U. Fugacci, F. Iuricich. ***Homological shape analysis through discrete Morse theory.*** Perspectives in Shape Analysis, pages 187-209, 2016.